ANALYTICAL SOLUTION OF THE MHD PROBLEM TO THE FLOW OVER THE ROUGHNESS ELEMENTS USING THE DIRAC DELTA FUNCTION

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INTRODUCTION

In the designing of the present reactor tokamak the value of the Hartmann boundary layer in a strong magnetic field becomes commensurable with the size of the roughness of the surface of a channel's wall. Therefore, it is need to study the influence of the roughness of the surface on the MHD flow of the conducting metal, which is planed to use in the system of the cooling of the reactor.

In paper [1] it is solved the MHD problem on the flow of conducting fluid in the half space, arising in the consequence of the roughness of the surface in the form $\tilde{z} = \tilde{\chi}_0 \tilde{f}(\tilde{x})$ with the conditions that the values $\left|\tilde{f}(\tilde{x})\right|$ and $\left|\tilde{f}'(\tilde{x})\right|$ are small. These assumptions give opportunity to transfer the boundary condition for potential of current $\tilde{\Phi}(\tilde{x},\tilde{z})$ from surface $\tilde{z} = \tilde{\chi}_0 \tilde{f}(\tilde{x})$ to plane $\tilde{z} = 0$ and neglect in this boundary condition term $\tilde{f}'(\tilde{x})\partial\tilde{\Phi}(\tilde{x},0)/\partial\tilde{x}$. The attempt takes into account that term lead to integral equation for unknown function $\partial\tilde{\Phi}(\tilde{x},0)/\partial\tilde{x}$, which one can solve only numerically. In paper [2] this problem is solved for the case when the roughness of surface $\tilde{z} = \tilde{\chi}_0 \tilde{f}(\tilde{x})$ has the rectangular form: $\tilde{z} = \tilde{\chi}_0$, if $\tilde{x} \in (-L, L)$ and $\tilde{z} = 0$, if $\tilde{x} \notin [-L, L]$. As a result the derivative $\tilde{f}'(\tilde{x})$ in boundary condition is expressed through the Dirac delta function and instead of integral equation for function $\tilde{f}'(\tilde{x})\partial\tilde{\Phi}(\tilde{x},0)/\partial\tilde{x}$ is appeared unknown constant $\partial\tilde{\Phi}(L,0)/\partial\tilde{x}$. The last is permitted to solve this problem analytically and estimate the error, which give the neglect of term $\tilde{f}'(\tilde{x})\partial\tilde{\Phi}(\tilde{x},0)/\partial\tilde{x}$ in mentioned above boundary condition. Besides the asymptotic of this problem in a strong magnetic field is obtained. In this paper similar problem for the constant cross-section of prism bounded by step-function form is solved.

1 THE STATEMENT OF THE PROBLEM

The geometry of the flow, which was considered in author's paper [2], is shown on Fig.1. The conducting fluid is located in the half space $\tilde{z} > 0$, $-\infty < \tilde{x}, \tilde{y} < +\infty$. The external magnetic field has the form

$$\mathbf{B}^{e} = \mathrm{B}_0 \mathbf{e}_z . \qquad (1.1)$$

The boundary $\tilde{z} = 0$ is not conducting. A steady current flows with the density $\tilde{\mathbf{j}} = j_0 \mathbf{e}_x$ in the direction of the x-axis. If the surface $\tilde{z} = 0$ is ideally smooth then the flow is absent because electromagnetic force $\vec{F} = \vec{j} \times \vec{B}$ is constant and $rot\vec{F} = 0$. In paper [2] the roughness of the surface $\tilde{z} = 0$ with cross section in the form of rectangular has been used (see Fig.1):

$$\tilde{z} = \tilde{\chi}_0 \tilde{f}(\tilde{x}) = \tilde{\chi}_0[\eta(\tilde{x}+L) - \eta(\tilde{x}-L)] = \begin{cases} \tilde{\chi}_0, & -L < \tilde{x} < L, \\ 0, & |\tilde{x}| > L, \end{cases} \tag{1.2}$$

where $\eta(\tilde{x})$ is the Heaviside step function: $\eta(\tilde{x}) = \begin{cases} 0, & \tilde{x} < 0, \\ 1, & \tilde{x} > 0. \end{cases}$ (1.3)

In this case the full current is equal to $\mathbf{j} = \mathbf{j}_0 + \mathbf{j}(\tilde{x}, \tilde{z})$ and the flow of the fluid with the velocity $\mathbf{V} = \tilde{V}_y(\tilde{x}, \tilde{z})\mathbf{e}_y$ arises in the direction opposite to the $\tilde{y}$ axis (Fig.1).

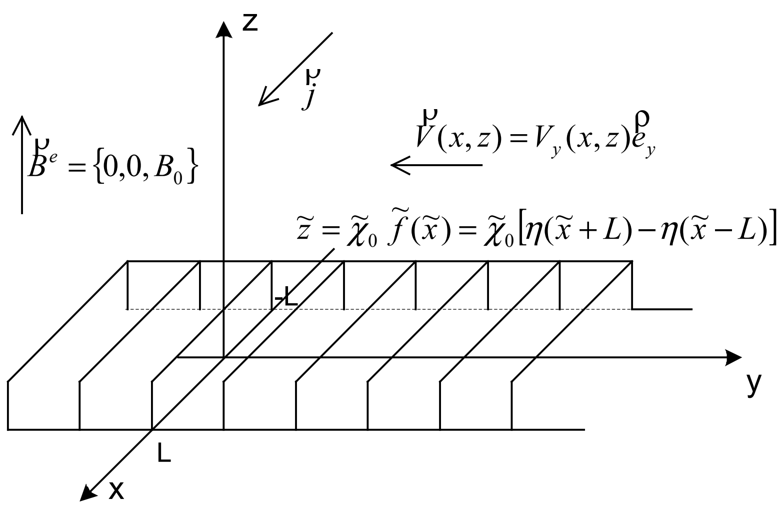

Figure 1. The geometry of the flow in paper [2].

In this paper we consider the similar problem with the constant cross-section in the form of the step-function (see Fig.2):

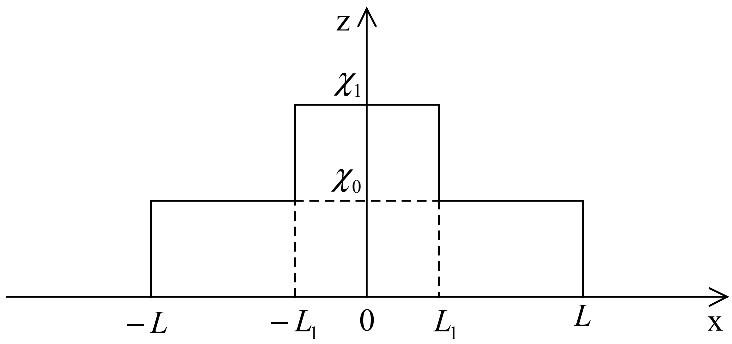

Figure 2. The constant cross-section of the roughness in this paper.

$$\tilde{z} = \tilde{F}(\tilde{x}) = \begin{cases} \tilde{\chi}_1, & |\tilde{x}| < L_1 \\ \tilde{\chi}_0, & L_1 < |\tilde{x}| < L \\ 0, & |\tilde{x}| > L \end{cases} \tag{1.4}$$

or $\tilde{z} = \tilde{F}(\tilde{x}) = \tilde{\chi}_0\, \tilde{f}_1(\tilde{x}) + (\tilde{\chi}_1 - \tilde{\chi}_0)\, \tilde{f}_2(\tilde{x})$, (1.5)

where

$$\tilde{f}_1(\tilde{x}) = \eta(\tilde{x} + L) - \eta(\tilde{x} - L), \quad \tilde{f}_2(\tilde{x}) = \eta(\tilde{x} + \tilde{L}_1) - \eta(\tilde{x} - \tilde{L}_1). \tag{1.6}$$

We will deduce the boundary condition for the potential $\tilde{\Phi}(\tilde{x}, \tilde{z})$ of an electrical field on the surface $\tilde{z} = \tilde{F}(\tilde{x})$. The normal component of the current on this surface must be equal to zero because the boundary $\tilde{z} = \tilde{\chi}_0 \tilde{f}(\tilde{x})$ is not conducting, i.e. must be $\mathbf{j} \cdot \mathbf{n} = 0$ on the surface ($\mathbf{n}$ is the unit vector of the normal to the surface).

Using formula $\mathbf{n} = grad[\tilde{z} - \tilde{F}(\tilde{x})] / \sqrt{1 + \tilde{F}^2(\tilde{x})}$ we obtain

$$\mathbf{n} = \left[\tilde{F}'(\tilde{x})\mathbf{e}_x + \mathbf{e}_z\right] / \sqrt{1 + \tilde{F}'^2(\tilde{x})}, \tag{1.7}$$

where

$$\tilde{F}'(\tilde{x}) = \tilde{\chi}_0[\delta(\tilde{x} + L) - \delta(\tilde{x} - L)] + (\tilde{\chi}_1 - \tilde{\chi}_0)\left[\delta(\tilde{x} + \tilde{L}_1) - \delta(\tilde{x} - \tilde{L}_1)\right], \tag{1.8}$$

$\delta(\tilde{x})$ is the Dirac delta function.

Putting the $\mathbf{n}$ from (1.7) and $\tilde{\mathbf{j}} = \left(j_0 + \tilde{j}_x(\tilde{x}, \tilde{z})\right)\mathbf{e}_x + \tilde{j}_z(\tilde{x}, \tilde{z})\mathbf{e}_z$ into $\tilde{\mathbf{j}} \cdot \mathbf{n} = 0$ and using formula $\tilde{\mathbf{j}} = \sigma\left[-grad\tilde{\mathbf{\Phi}} + \tilde{\mathbf{V}} \times \mathbf{B}\right]$, i.e. $\tilde{j}_x = -\sigma\, \partial\tilde{\Phi} / \partial\tilde{x}$, $\tilde{j}_z = -\sigma\, \partial\tilde{\Phi} / \partial\tilde{z}$ on the surface, where $\tilde{\mathbf{V}} = 0$, we obtain the boundary condition for the potential $\tilde{\Phi}(\tilde{x}, \tilde{z})$:

$$\tilde{z} = \tilde{F}(\tilde{x}): \quad -\sigma \frac{\partial\tilde{\Phi}}{\partial\tilde{z}} = \tilde{F}'(\tilde{x}) \left[j_0 - \sigma \frac{\partial\tilde{\Phi}}{\partial\tilde{x}} \right], \tag{1.9}$$

where function $\tilde{F}'(\tilde{x})$ gives formula (1.8).

We do in this paper the single approximation: we transfer the boundary condition (1.9) from the surface $\tilde{z} = \tilde{F}(\tilde{x})$ to the plane $\tilde{z} = 0$, i.e. we suppose that only the value $\left|\tilde{F}(\tilde{x})\right|$ is small. As a result, we obtain the boundary condition for the potential in the form

$$\tilde{z} = 0: \quad \partial\tilde{\Phi} / \partial\tilde{z} = \left[-j_0\sigma^{-1} + \partial\tilde{\Phi} / \partial\tilde{x}\right]\tilde{F}'(\tilde{x}). \tag{1.10}$$

We don't neglect the term $\partial\tilde{\Phi} / \partial\tilde{x}$ in boundary condition (1.9) and as a result we obtain the new coefficient in the solution used in paper [1].

We use the following dimensionless quantities using the values L, ν / L, B_0, $\nu\sqrt{\rho\nu/\sigma}/L$, $\nu\sqrt{\rho\nu\sigma}/L^2$ as scales of length, velocity, magnetic field, potential and current, respectively. Here σ, ρ, ν are, respectively, the conductivity, the density and the viscosity of the fluid. Then the MHD equations and the boundary conditions have the form (see [3]):

$$\Delta V_y - Ha^2 V_y + Ha \cdot \partial\Phi/\partial x = 0, \quad \Delta\Phi = Ha \cdot \partial V_y/\partial x, \tag{1.11,12}$$

$$z = 0: V_y = 0, \partial\Phi/\partial z = [-A + F(x,0)]F'(x), \tag{1,13,14}$$

$$F'(x) = \chi_0[\delta(x+1) - \delta(x+1)] + (\chi_1 - \chi_0)[\delta(x + L_1) - \delta(x - L_1)], \tag{1.15}$$

$$\sqrt{x^2 + z^2} \to \infty: \quad V_y \to 0, \Phi \to 0, \tag{1.16}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$, $Ha = B_0 L\sqrt{\sigma/\rho\nu}$ is the Hartmann number, $A = j_0 L^2/(\nu\sqrt{\rho\nu\sigma})$, $\chi_0 = \tilde{\chi}_0/L$, $\chi_1 = \tilde{\chi}_1/L$ and

$$F(x,0) = \left.\frac{\partial\Phi}{\partial x}\right|_{z=0}. \tag{1.17}$$

2 THE SOLUTION OF PROBLEM (1.11)-(1.16)

In order to solve problem (1.11)-(1.16) we use the symmetry of this problem with respect to x: the function $V_y(x,z)$ is an even function, $\Phi(x,z)$ is an odd function with respect to x. This means that functions $V_y(x,z)$ and $\Phi(x,z)$ satisfy additional boundary conditions:

$$z = 0: \quad \frac{\partial V_y}{\partial x} = 0, \Phi(x,0) = 0. \tag{2.1}$$

Therefore problem (1.11)-(1.16) can be solved by means of Fourier cosine and Fourier sine transforms (see[4]). Namely, we apply the Fourier cosine transform with respect to x to equation (1.11) and to V_y in boundary condition (1.13) and the Fourier sine transform to equation (1.12) and to $\partial\Phi/\partial z$ in boundary condition (1.14), putting:

$$V_y^c(\lambda, z) = \sqrt{\frac{2}{\pi}}\int_0^\infty V_y(x,z)\cos\lambda x dx, \tag{2.2}$$

$$\Phi^s(\lambda, z) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Phi(x,z)\sin\lambda x dx. \tag{2.3}$$

As a result we obtain the following system of ordinary differential equations for unknown functions $V_y^{\,c}(\lambda,z)$, $\Phi^s(\lambda,z)$:

$$-\lambda^2 V_y + \frac{d^2 V_y^{\,c}}{dz^2} - Ha^2 V_y^{\,c} + Ha\lambda\Phi^s = 0, \tag{2.4}$$

$$-\lambda^2 \Phi^s + \frac{d^2\Phi^s}{dz^2} + Ha\lambda V_y^{\,c} = 0. \tag{2.5}$$

We apply also transforms (2.2) and (2.3) to boundary conditions (1.13) and (1.14):

$$z=0:\ \ V_y^{\,c} = 0,\ \ \frac{d\Phi^s}{dz} = \sqrt{2\pi}D_1\sin\lambda + \sqrt{2\pi}D_2\sin\lambda L_{1,}\,;\ z\to\infty: V_y^{\,c}, \Phi^s \to 0, \tag{2.6,7}$$

where $D_1 = \dfrac{\chi_0}{\pi}[A - F(1,0)], D_2 = \dfrac{\chi_1-\chi_0}{\pi}[A - F(L_1,0),$ (2.8)

$$F(1,0) = \frac{\partial\Phi}{\partial x}\ \text{ at }\ x=1,\ z=0,\ F(L_1,0) = \frac{\partial\Phi}{\partial x}\ \text{ at }\ x=L_1,\ z=0 \tag{2.9}$$

are the unknown constans. The solution of the problem (2.4)-(2.7) has the form:

$$\Phi^s(\lambda,z) = \frac{1}{2\lambda^2}\left(k_1e^{k_2z} + k_2e^{k_1z}\right)\ [\sqrt{2\pi}D_1\sin\lambda + \sqrt{2\pi}D_2\sin\lambda L_{1,}], \tag{2.10}$$

$$V^c(\lambda,z) = \frac{1}{2\lambda}\left(e^{k_1z} - e^{k_2z}\right)[\sqrt{2\pi}D_1\sin\lambda + \sqrt{2\pi}D_2\sin\lambda L_{1,}], \tag{2.11}$$

where

$$k_1 = -(\sqrt{\lambda^2+\mu^2}+\mu),\ \ k_2 = -(\sqrt{\lambda^2+\mu^2}-\mu),\ \ 2\mu = Ha. \tag{2.12}$$

Applying to formulae (2.10), (2.11) the inverse Fourier sine and cosine transforms, we obtain the solution of problem (1.11)-(1.16), containing unknown constants $F(1,0)$, $F(L_1,0)$:

$$\Phi(x,z) = D_1\int_0^\infty \left(k_1e^{k_2z} + k_2e^{k_1z}\right)\frac{\sin\lambda}{\lambda^2}\sin\lambda x d\lambda +$$

$$+ D_2\int_0^\infty \left(k_1e^{k_2z} + k_2e^{k_1z}\right)\frac{\sin\lambda L_1}{\lambda^2}\sin\lambda x d\lambda, \tag{2.13}$$

$$V_y(x,z) = D_1\int_0^\infty \left(e^{k_1z} - e^{k_2z}\right)\frac{\sin\lambda}{\lambda}\cos\lambda x d\lambda +$$

$$+D_2\int_0^\infty \left(e^{k_1 z}-e^{k_2 z}\right)\frac{\sin\lambda L_1}{\lambda}\cos\lambda x d\lambda. \tag{2.14}$$

The components j_x and j_z of the induced current density we obtain using law:

$$\tilde{\mathbf{j}}=\sigma\left[-grad\tilde{\Phi}(\tilde{x},\tilde{z})+\tilde{\mathbf{V}}\times\tilde{\mathbf{B}}\right], \tag{2.15}$$

where $\tilde{\mathbf{V}}=\tilde{V}_y(\tilde{x},\tilde{z})\mathbf{e}_y$, $\tilde{\mathbf{B}}=\tilde{B}_y^i(\tilde{x},\tilde{z})\mathbf{e_y}+B_0\mathbf{e_z}$. (2.16)

In the dimensionless quantities formula (2.15) has the form

$$\mathbf{j}=-grad\Phi(x,z)+Ha\mathbf{V}\times\mathbf{B}, \tag{2.17}$$

where $\mathbf{V}=V_y(x,z)\mathbf{e}_y$, $\mathbf{B}=B_y^i(x,z)\mathbf{e_y}+\mathbf{e_z}$. (2.18)

Substituting (2.18) into (2.17) one gives

$$\mathbf{j}=-grad\Phi(x,z)+HaV_y(x,z)\mathbf{e_x}. \tag{2.19}$$

From (2.18) it follows, that

$$j_x=-\frac{\partial\Phi}{\partial x}+HaV_y(x,z),\quad j_z=-\frac{\partial\Phi}{\partial z} \tag{2.20}$$

or, using formulae (2.12), (2.13) and (2.14),

$$j_x=-D_1\int_0^\infty\left(k_1e^{k_1z}+k_2e^{k_2z}\right)\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda-D_2\int_0^\infty\left(k_1e^{k_1z}+k_2e^{k_2z}\right)\frac{\sin\lambda L_1\cos\lambda x}{\lambda}d\lambda, \tag{2.21}$$

$$j_z=-D_1\int_0^\infty\left(e^{k_1z}+e^{k_2z}\right)\sin\lambda\sin\lambda x d\lambda-D_2\int_0^\infty\left(e^{k_1z}+e^{k_2z}\right)\sin\lambda L_1\sin\lambda x d\lambda. \tag{2.22}$$

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For the evaluation of unknown constants $F(1,0), F(L_1,0)$ or D_1, D_2 in formulae (2.13), (2.14), (2.21) and (2.22) it is necessary to use integral (2.13) and evaluate the limit

$$F(1,0)=D_1\lim_{z\to+0}\int_0^\infty\left(k_1e^{k_2z}+k_2e^{k_1z}\right)\frac{\sin\lambda\cos\lambda}{\lambda}d\lambda+D_2\lim_{z\to+0}\int_0^\infty\left(k_1e^{k_2z}+k_2e^{k_1z}\right)\frac{\sin\lambda L_1\cos\lambda}{\lambda}d\lambda \tag{2.23}$$

and similar limit for $F(L_1,0)$. Differentiation with respect to x under the integral singe in (2.13) is correct in the region $0 < z_0 \le z < +\infty, 0 \le x < +\infty$ because this integral, also as corresponding integral (2.23) of partial derivative with respect to x of integrand in (2.13) is majorized in this region. However, if we put $z = 0$ under integral sign in (2.23), we obtain the divergent integral, which is converged only in the sense of Abel (see [4]). For example, for the first integral in the right hand side in (2,23) we obtain:

$$I \equiv \int_0^\infty \sqrt{\lambda^2+\mu^2}\,\frac{\sin 2\lambda}{\lambda}\,d\lambda = \lim_{\delta\to+0}\int_0^\infty e^{-\delta\lambda}\sqrt{\lambda^2+\mu^2}\,\frac{\sin 2\lambda}{\lambda}\,d\lambda \tag{2.24}$$

or, after evident transformations

$$I \equiv \lim_{\delta\to+0}\int_0^\infty e^{-\delta\lambda}\frac{\mu^2}{\sqrt{\lambda^2+\mu^2}+\lambda}\frac{\sin 2\lambda}{\lambda}\,d\lambda + \lim_{\delta\to+0}\int_0^\infty e^{-\delta\lambda}\sin 2\lambda d\lambda. \tag{2.25}$$

The first integral in the right hand side of (2.25) converge in the usual sense, but the second integral converge only in the sense of Abel and equal to ½ (see [4]). However such method gives the solution, which tends to zero as Hartmann number Ha tends to infinity. The last contradicts to the physical sense of problem. Therefore it is need to transform integral (2.13) to such form that after passing to limit $z \to +0$ we would obtain the integral, converging in the usual sense. For this purpose we use formulae:

$$\int_0^\infty e^{-z\sqrt{\lambda^2+\mu^2}}\cos a\lambda d\lambda = \frac{\mu z}{\sqrt{z^2+a^2}}K_1\left(\mu\sqrt{z^2+a^2}\right), \tag{2.26}$$

$$\int_0^\infty \sqrt{\lambda^2+\mu^2}\,e^{-z\sqrt{\lambda^2+\mu^2}}\cos a\lambda d\lambda = \frac{\mu}{\sqrt{z^2+a^2}}\left[\frac{\mu z^2}{\sqrt{z^2+a^2}}K_2\left(\mu\sqrt{z^2+a^2}\right) - K_1\left(\mu\sqrt{z^2+a^2}\right)\right], \tag{2.27}$$

where $a \ge 0$, $z > 0$ and $K_\nu(z)$ is the modified Bessel function of the second kind of order ν (ν=1, 2). As a result, we obtain (the details see in [5]):

$$V_y(x,z) = -\mu z\cdot sh\mu z\left[D_1\int_{x-1}^{x+1}\frac{K_1\left(\mu\sqrt{z^2+t^2}\right)}{\sqrt{z^2+t^2}}dt + D_2\int_{x-L_1}^{x+L_1}\frac{K_1\left(\mu\sqrt{z^2+t^2}\right)}{\sqrt{z^2+t^2}}dt\right] \tag{2.28}$$

$$j_x(x,z) = ch\mu z\{D_1[F(1+x)-F(1-x)] + D_2[F(L_1+x)-F(L_1-x)]\} + \mu V_y(x,z), \tag{2.29}$$

where

$$F(a) = \int_0^a \frac{\mu}{\sqrt{z^2+t^2}}\left[\frac{\mu z^2}{\sqrt{z^2+t^2}}K_2\left(\mu\sqrt{z^2+t^2}\right) - K_1\left(\mu\sqrt{z^2+t^2}\right)\right]dt. \tag{2.30}$$

The evaluation of integral (2.22) gives:

$$j_z(x,z) = \mu z \cdot ch\mu z\left[D_1 G(x,z,1) + D_2 G(x,z,L_1)\right], \tag{2.31}$$

where

$$G(x,z,L_1) = \frac{K_1\left(\mu\sqrt{z^2+(L_1-x)^2}\right)}{\sqrt{z^2+(L_1-x)^2}} - \frac{K_1\left(\mu\sqrt{z^2+(L_1+x)^2}\right)}{\sqrt{z^2+(L_1+x)^2}}. \tag{2.32}$$

We transform ∂Φ/∂x, using formulae (2.13), (2.26), (2.27):

$$\left.\frac{\partial\Phi}{\partial x}\right|_{x=1} = -D_1\left\{ch\mu z\int_0^2 \frac{\mu}{\sqrt{z^2+t^2}}\left[\frac{\mu z^2}{\sqrt{z^2+t^2}}K_2\left(\mu\sqrt{z^2+t^2}\right) - K_1\left(\mu\sqrt{z^2+t^2}\right)\right]dt + \right.$$

$$\left. + \mu^2 z\cdot sh\mu z\int_0^2 \frac{1}{\sqrt{z^2+t^2}}K_1\left(\mu\sqrt{z^2+t^2}\right)dt\right\} -$$

$$-D_2\left\{ch\mu z\int_{L_1-1}^{L_1+1} \frac{\mu}{\sqrt{z^2+t^2}}\left[\frac{\mu z^2}{\sqrt{z^2+t^2}}K_2\left(\mu\sqrt{z^2+t^2}\right) - K_1\left(\mu\sqrt{z^2+t^2}\right)\right]dt + \right.$$

$$\left. + \mu^2 z\cdot sh\mu z\int_{L_1-1}^{L_1+1} \frac{1}{\sqrt{z^2+t^2}}K_1\left(\mu\sqrt{z^2+t^2}\right)dt\right\}. \tag{2.33}$$

The integrals in the right hand side of formula (2.33) are diverged if $z = 0$. Therefore we use the substitution

$$t = z\xi,\quad dt = zd\xi. \tag{2.34}$$

Then from formula (2.33) follow

$$\left.\frac{\partial\Phi}{\partial x}\right|_{x=1} = -D_1\left\{ch\mu z\int_0^{\frac{2}{z}} \frac{\mu}{\sqrt{1+\xi^2}}\left[\frac{\mu z}{\sqrt{1+\xi^2}}K_2\left(\mu z\sqrt{1+\xi^2}\right) - K_1\left(\mu z\sqrt{1+\xi^2}\right)\right]d\xi + \right.$$

$$\left. + \mu\cdot sh\mu z\int_0^{\frac{2}{z}} \frac{\mu z}{\sqrt{1+\xi^2}}K_1\left(\mu z\sqrt{1+\xi^2}\right)d\xi\right\} -$$

$$-D_2\left\{ch\mu z\int_{\frac{L_1-1}{z}}^{\frac{L_1+1}{z}} \frac{\mu}{\sqrt{1+\xi^2}}\left[\frac{\mu z}{\sqrt{1+\xi^2}}K_2\left(\mu z\sqrt{1+\xi^2}\right) - K_1\left(\mu z\sqrt{1+\xi^2}\right)\right]d\xi\right.$$

$$\left. + \mu\cdot sh\mu z\int_{\frac{L_1-1}{z}}^{\frac{L_1+1}{z}} \frac{\mu z}{\sqrt{1+\xi^2}}K_1\left(\mu z\sqrt{1+\xi^2}\right)d\xi\right\}. \tag{2.35}$$

For pass to limit at $z \to +0$ in formula (2.34) we use formula

$$K_n(z) \approx \frac{1}{2}(n-1)!\left(\frac{2}{z}\right)^n, \ n = 1,2,3,... \text{ at } z \to +0,$$

i.e. $K_1(z) \approx \frac{1}{z}$, $K_2(z) \approx \frac{2}{z^2}$ at $z \to +0$. (2.36)

As a result we obtain from formula (2.35) that

$$\lim_{z\to+0}\frac{\partial\Phi}{\partial x}\bigg|_{x=1} = -D_1\lim_{z\to+0}\frac{1}{z}\int_0^{\frac{2}{z}}\left[\frac{2}{\left(1+\xi^2\right)^2}-\frac{1}{1+\xi^2}\right]d\xi - D_1\lim_{z\to+0}\mu\cdot sh\mu z\int_0^{\frac{2}{z}}\frac{1}{1+\xi^2}d\xi -$$

$$-D_2\lim_{z\to+0}\frac{1}{z}\int_{\frac{L_1-1}{z}}^{\frac{L_1+1}{z}}\left[\frac{2}{\left(1+\xi^2\right)^2}-\frac{1}{1+\xi^2}\right]d\xi - D_2\lim_{z\to+0}\mu\cdot sh\mu z\int_{\frac{L_1-1}{z}}^{\frac{L_1+1}{z}}\frac{1}{1+\xi^2}d\xi. \quad (2.37)$$

The second and the last limits in the right hand side of formula (2.37) are equal to zero, but the first and the third limits gives indefiniteness of the form $\frac{0}{0}$ because

$$\int_0^{\infty}\left[\frac{2}{\left(1+\xi^2\right)^2}-\frac{1}{1+\xi^2}\right]d\xi = \int_0^{\infty}\frac{2}{\left(1+\xi^2\right)^2}d\xi - \frac{\pi}{2} = 0. \quad (2.38)$$

Consequently, from formula (2.37) we obtain

$$\lim_{z\to+0}\frac{\partial\Phi}{\partial x}\bigg|_{x=1} = -D_1\lim_{z\to+0}\frac{1}{z}\int_0^{\frac{2}{z}}\left[\frac{2}{\left(1+\xi^2\right)^2}-\frac{1}{1+\xi^2}\right]d\xi - D_2\lim_{z\to+0}\frac{1}{z}\int_{\frac{L_1-1}{z}}^{\frac{L_1+1}{z}}\left[\frac{2}{\left(1+\xi^2\right)^2}-\frac{1}{1+\xi^2}\right]d\xi =$$

$$= -D_1\lim_{z\to+0}\left[\frac{2}{\left(1+\frac{4}{z^2}\right)^2}-\frac{1}{1+\frac{4}{z^2}}\right]\left(-\frac{2}{z^2}\right) - D_2\lim_{z\to+0}\left[\frac{2}{\left(1+\frac{(L_1+1)^2}{z^2}\right)^2}-\frac{1}{1+\frac{(L_1+1)^2}{z^2}}\right]\left(-\frac{L_1+1}{z^2}\right)+$$

$$+D_2\lim_{z\to+0}\left[\frac{2}{\left(1+\frac{(L_1-1)^2}{z^2}\right)^2}-\frac{1}{1+\frac{(L_1-1)^2}{z^2}}\right]\left(-\frac{L_1-1}{z^2}\right) = -\frac{D_1}{2}+\frac{2D_2}{1-L_1^2}. \quad (2.39)$$

From (2.8) and (2.39) it follows

$$F(1,0) = -\frac{1}{2}D + \frac{2D_2}{1-L_1^{\ 2}}. \tag{2.40}$$

Similarly for $F(L_1,0)$ we obtain:

$$F(L_1,0) = \frac{2D_1}{1-L_1^{\ 2}} - \frac{D_2}{2L_1}. \tag{2.41}$$

We remind (see formula (2.8), that

$$D_1 = \frac{\chi_0}{\pi}[A - F(1,0)],\ \ D_2 = \frac{\chi_1 - \chi_0}{\pi}[A - F(L_1,0)]. \tag{2.42}$$

Consequently formulae (2.40) and (2.41) are the system of two equations for two unknown constants $F(1,0)$ and $F(L_1,0)$, i.e. for two unknown constants D_1 and D_2.

Substituting these constants into formulae (2.13), (2.14) and (2.21), (2.22), we obtain the solution of problem (1.11)-(1.16).

3 THE ASYMPTOTIC EVALUATION OF THE PROBLEM AND THE NUMERICAL RESULTS FOR THE CASE $\chi_1 = \chi_0$

The asymptotic of functions $V_y(x,z)$, $j_x(x,z)$ $j_z(x,z)$ we get from integrals (2.14), (2.21) and (2.22), using the formula, which holds at $\mu \to \infty$:

$$-k_1 = \left(\sqrt{\lambda^2+\mu^2} - \mu\right) + 2\mu = 2\mu + \frac{\lambda^2}{\sqrt{\lambda^2+\mu^2}+\mu} \approx 2\mu + \frac{\lambda^2}{2\mu}, \tag{3.1}$$

$$-k_2 = \sqrt{\lambda^2+\mu^2} - \mu \approx \frac{\lambda^2}{2\mu},\ \ 2\mu = Ha. \tag{3.2}$$

Substituting (3.1) and (3.2) into integrals (2.14), (2.21), (2.22) and using the Poisson integral (see [6]), we obtain the asymptotic formulae, which holds for the whole region $0 < z < +\infty$ as $Ha \to \infty$:

$$V_y(x,z) = -\frac{\pi}{4}D_1\left(1 - e^{-zHa}\right)\psi(x,z), \tag{3.3}$$

$$\psi(x,z) = erf\left[\beta\frac{1+x}{\sqrt{z}}\right] + erf\left[\beta\frac{1-x}{\sqrt{z}}\right],\ \ \beta = 0.5\sqrt{Ha}, \tag{3.4}$$

$$j_x = D_1\left\{\frac{\pi}{4}Hae^{-zHa}\psi(x,z)+\frac{\beta}{z\sqrt{\pi z}}\left(1+\frac{\pi}{4}e^{-zHa}\right)\left[(1+x)e^{\frac{-\beta^2(1+x)^2}{z}}+(1-x)e^{\frac{-\beta^2(1-x)^2}{z}}\right]\right\},$$

(3.5)

$$j_z = -\frac{1}{2}D_1\beta\sqrt{\frac{\pi}{z}\left(1+e^{-zHa}\right)}\left[e^{\frac{-\beta^2(1-x)^2}{z}}-e^{\frac{-\beta^2(1+x)^2}{z}}\right]. \qquad (3.6)$$

From formulae (3.3) and (3.6) it follows all the above-obtained results about boundary layers of functions $V_y(x,z)$ and $j_z(x,z)$ as $Ha \to \infty$.

We see from formula (3.3) that at $Ha \to \infty$ we have:

1) Component $V_y = \frac{-\pi}{2}D_1 = \text{constant}$ in region $Ha^{-1} < z < Ha$.

2) Component V_y is changed from 0 till $V_y = \frac{-\pi}{2}D_1 = V_c$ in region $0 < z < Ha^{-1}$.

3) Component V_y is changed from V_c till zero in region $Ha < z < +\infty$.

Besides from formula (3.3) it follows that on the lines $x = \pm 1$, $0 < z < +\infty$ component $V_y \to 0.5V_c$ as $Ha \to \infty$. That means that the two new boundary layers exist in the regions:

$$-\varepsilon < \beta\frac{1-x}{\sqrt{z}} < \varepsilon \text{ and } -\varepsilon < \beta\frac{1+x}{\sqrt{z}} < \varepsilon, \qquad (3.7)$$

where ε is some enough small positive number. In these regions component V_y is changed between $-V_c$ and zero. It is impossible to get these two new boundary layers from formula (2.14).

Similarly, we see from (3.6) that at $Ha \to \infty$:

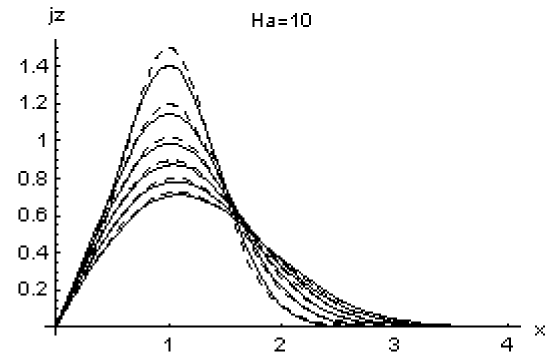
1. Component j_z exponentially tends to zero everywhere except for the two regions, lying inside parabolas

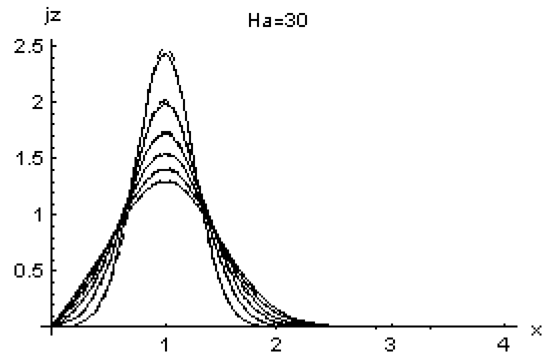
$$z = 0.5\mu(x+1)^2 \text{ and } z = 0.5\mu(x-1)^2. \qquad (3.8)$$

because in this case both the exponent in the square bracket of formula (3.6) tends to zero.
2. Inside the region bounded by the first or second parabola in (3.8), where one of the exponents in the square bracket of (3.6) does not equal to zero, component $j_z \approx \sqrt{\mu/z}$, i.e. tends to infinity as $\mu \to \infty$.
3. Finally, we see from formula (3.5) that at $Ha \to \infty$ the current component $j_x(x,z)$ tends to zero everywhere except for the region $0 < z < Ha^{-1}$ because in this region

$\exp(-zHa) \neq 0$ and the function $\psi(x,z)$ tends to 2 everywhere except for the two regions in formula (3.7).

For the evaluation of Hartmann numbers at which the asymptotic formulae (3.3)-(3.6) are correct we compare the numerical results for the component $j_z(x,z)$, obtained by exact formula (2.31) and by asymptotic formula (3.6). These numerical results for Hartmann numbers Ha =10, 30, 50 are shown on Fig.3 . For Hartmann numbers $Ha \geq 10$ the results obtained by exact formula (2.31) and by asymptotic formula (3.6) practically coincide. Calculations for functions $V_y(x,z)$ and $j_x(x,z)$ by exact formulae (2.14), (2.21) and by asymptotic formulae (3.3), (3.5) give coincidence at the same Hartmann numbers .





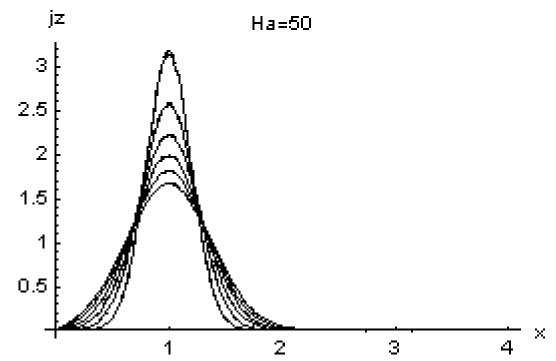
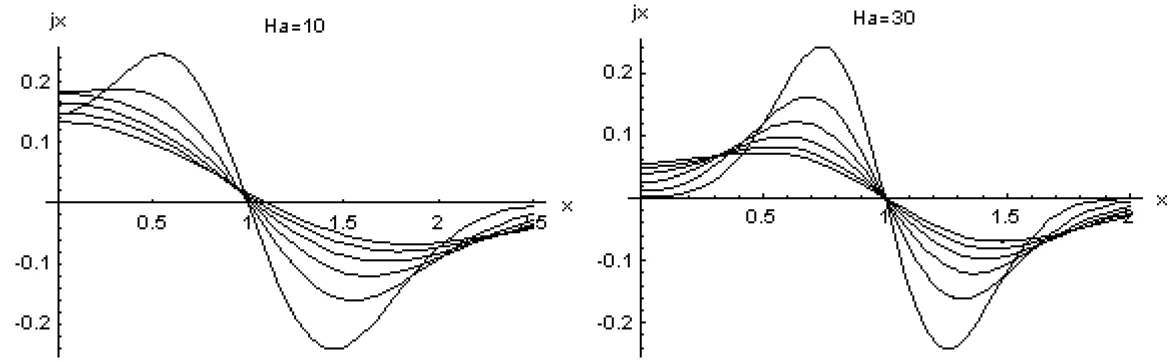


Figure 3. The graphics of the z-component of current by exact formula (2.31) and by formula (3.6) from z=1 (two upper lines) till z=3.5 (two lower lines) through $\Delta z = 0.5$. Function $j_z(x,z)$ is odd with respect to x.

On the Fig.4 are shown the numerical results of calculation of the current's component $j_x(x,z)$ by asymptotic formula (3.5) for Hartmann numbers Ha=10, 30 and 50. We can see, that the sign of function $j_x(x,z)$ is changed in the neighborhood of line $x = 1$, $0 < z < +\infty$. It means that the streamlines of current $\vec{j}(x,z)$ are changed their direction on the opposite in the neighborhood of this line.



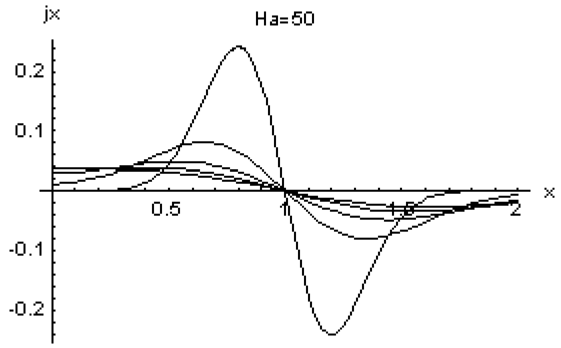


Figure 4. The graphics of the x-component of current by asymptotic formula (3.5) from z=1 (the upper lines in region $0 \le x \le 1$) till z=3.5 (the lower lines in region $0 \le x \le 1$) through $\Delta z = 0.5$. Function $j_x(x,z)$ is even with respect to x.

4 CONCLUSIONS

1. The analytical solution of the two dimensional problem on the MHD flow in half space $z \ge 0$ in the consequence of the roughness of the boundary of special form is obtained. The roughness with constant cross section, bounded by step-function, is located along the y axis. There are the external current which flows parallel to x axis and the external magnetic field parallel to z axis. The two dimensional MHD flow in the direction opposite to y axis arises, only if the roughness of the boundary is present.
2. The analytical solution is obtained at the single approximate assumption that the height of the roughness is small .The solutions for the y component of the velocity of the fluid and for the x component of the induced current are obtained in the form of improper integrals of elementary functions. On the other hand, the z component of the induced current is expressed through the Bessel function.
3. The asymptotic solution of the problem at Hartmann number $Ha \to \infty$ is obtained in the form of elementary functions. For Hartmann numbers Ha $\ge$ 10 the exact and the asymptotic solutions practically coincide.

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Antimirovs M., Čaddads I. Analītisks atrisinājums problēmai par MHD plūsmu pustelpā ar negludu robežvirsmu, izmantojot Dīraka delta funkciju.
Rakstā analītiski atrisināta problēma par MHD plūsmu bezgalīgā pustelpā $\tilde{z} > 0$*, kura satur speciālas formas negludumus uz robežas* $\tilde{z} = 0$*. Ārējais magnētiskais lauks darbojas perpendikulāri robežai* $\tilde{z} = 0$*. Darbojas arī ārējā strāva, kura ir paralēla robežai* $\tilde{z} = 0$*, ja robeža ir gluda. MHD plūsma rodas tikai tad, ja eksistē negludumi. Izvēloties negluduma formu prizmas veidā ar šķērsgriezuma laukumu, kurš ierobežots ar lēciena funkciju, izdevās iegūt iepriekš minētās problēmas analītisku atrisinājumu, izmantojot Dīraka delta funkciju. Tiek izmantots tikai viens pieņēmums, ka prizmas šķērsgriezuma augstums ir mazs. Darbā ir iegūts minētās problēmas analītisks atrisinājums pie lieliem Hartmaņa skaitļiem. Noteikti dažādi šķidruma plūsmas un inducētās strāvas robežslāņi. Veikti inducētās strāvas blīvuma vektora x un y komponenšu skaitliskie aprēķini.*

Antimirov M.Ya., Chaddad I.A. Analytical solution of the MHD problem to the flow over the roughness elements using the Dirac delta function.
Analytical solution of the problem about MHD flow of conducting fluid in half space $\tilde{z} > 0$ *with a roughness of special form on boundary* $\tilde{z} = 0$ *is obtained. External magnetic field is perpendicular to boundary* $\tilde{z} = 0$*. There is also external current, which is parallel to boundary* $\tilde{z} = 0$*, if the roughness is absent. The flow of fluid arises only in the case, if the roughness of boundary* $\tilde{z} = 0$ *exists. The choice of the roughness in the form of infinitely long prism with constantsr cross-section, bounded by the step-function, allows to obtain the analytical solution of this problem using the Dirac delta function. The single approximate assumption that the height of this cross-section is small is used. The asymptotic solution of the problem at the large Hartmann numbers is obtained. In this case the various boundary layers of the flow and the induced current are found. The results of numerical calculations of x- and y-component of induced current are present.*

Антимиров М.Я., Чаддад И.А. Аналитическое решение задачи об МГД течении по элементам шероховатости с помощью дельта функции Дирака.
Получено аналитическое решение задачи об МГД течении в полупространстве $\tilde{z} > 0$ *с элементами шероховатости специальной формы на границе* $\tilde{z} = 0$*. Внешнее магнитное поле перпендикулярно границе* $\tilde{z} = 0$*. Задан также внешний ток, который параллелен границе* $\tilde{z} = 0$*, если шероховатость отсутствует. МГД течение возникает только при наличии шероховатости. Выбор шероховатости в виде призмы с сечением, ограниченном ступенчатой функцией, позволил получить аналитическое решение этой задачи с помощью дельта функции Дирака. Использовано единственное приближенное предположение, что высота сечения призмы является малой. Получено также аналитическое решение данной задачи при больших числах Гартмана. Обнаружены различные пограничные слои для течения жидкости и для индуцированного тока. Приведены результаты числовых расчетов x- и y- компонент вектора плотности индуцированного тока.*