SCIENTIFIC JOURNAL OF RIGA TECHNICAL UNIVERSITY
Series - Computer Science - Boundary Field Problems and Computer Simulation - 51st thematic issue 2009

IMPEDANCE OF A COIL ABOVE A HALF-SPACE WITH VARYING ELECTRIC AND MAGNETIC PROPERTIES

SPOLES PRETESTĪBA VIRS ELEKTRISKI VADOŠAS PUSTELPAS AR MAINĪGĀM ELEKTRISKĀM UN MAGNĒTISKĀM ĪPAŠĪBĀM

V. Koliskina, I. Volodko

Keywords: eddy current testing, Hankel transform, Bessel functions

A two-parameter family of analytical solutions is found in the paper for the case where a single-turn coil with alternating current is located above an electrically conducting half-space. The electrical conductivity and magnetic permeability of the half-space are exponential functions of the vertical coordinate. The problem is solved by the method of Hankel integral transform. The solution is obtained in closed form in terms of improper integrals containing Bessel functions. Results of numerical calculations are presented. The obtained solution is also generalized for the case of a coil of finite dimensions. The formulas obtained in the paper can be used to solve the inverse problem of determining the parameters of a conducting half-space in cases where the electrical conductivity and magnetic permeability of the medium are not constant.

Introduction

In many engineering applications the external magnetic field can modify the electric and magnetic properties of the conducting material. Examples include surface hardening, decarbonization, surface alloying and determination of thickness of metal coatings [1], [2]. The changes in the electric and magnetic properties of the material can be taken into account by considering the solution obtained, for example, by Dodd and Deeds [3] for the case of a multilayer medium with constant properties. Up to 50 layers of a multilayer medium with constant properties were used in [4] to model the variation of the electric conductivity and magnetic permeability in the vertical direction. Alternatively, one can approximate the magnetic permeability and/or electric conductivity by continuously varying profiles of a relatively simple form for which the change in impedance of a coil can be found in closed form by means of known special functions [5]-[7]. In particular, analytical solution of two problems where either electric conductivity or magnetic permeability is exponentially varying with depth is considered in [5]. In the present paper we generalize the results of [5] for the case where both electric conductivity and magnetic permeability are exponential functions of the vertical coordinate. In addition, the formulas for the change in impedance are obtained not only for the case of a single-turn coil (as in [5]) but also for the case of a coil of finite dimensions.

Single-turn coil above a half-space with depth-varying electric and magnetic properties

Consider a single-turn coil of radius r_c situated at height h above a conducting half-space with electric conductivity σ and magnetic permeability μ. We assume that both σ and μ are exponentially varying functions of the vertical coordinate, namely,

$$\sigma = \sigma_m e^{\alpha z}, \mu = \mu_0 \mu_m e^{\beta z}, \tag{1}$$

where $\sigma_m, \mu_0, \mu_m, \alpha$ and β are constants.

Suppose that the vector potential $\vec{A}$ has only one nonzero component of the form

$$\vec{A} = A(r,z)\vec{e}_\varphi, \tag{2}$$

where $\vec{e}_\varphi$ is a unit vector in the φ-direction.

Here (r,φ,z) is a system of cylindrical polar coordinates centered at the origin, with the z-axis directed upwards. The alternating current in the coil is given by

$$i(t)\vec{e}_\varphi = I\exp(j\omega t)\vec{e}_\varphi,$$

where $j=\sqrt{-1}$, ω is the frequency and I is the amplitude of the current.

In this case the amplitude A of the vector potential satisfies the following system of equations (see [7]):

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I\delta(r-r_c)\delta(z-h), \tag{3}$$

$$\frac{\partial^2 A_1}{\partial r^2}+\frac{1}{r}\frac{\partial A_1}{\partial r}-\frac{A_1}{r^2}+\frac{\partial^2 A_1}{\partial z^2}-\beta\frac{\partial A_1}{\partial z}- \qquad (4)$$
$$-j\omega\sigma_m\mu_0\mu_m e^{(\alpha+\beta)z}A_1=0,$$

where $\delta(x)$ is the Dirac delta-function and the functions A_0 and A_1 represent the solutions in the regions $z>0$ and $z<0$, respectively.

The boundary conditions are

$$A_0\,|_{z=0}=A_1\,|_{z=0}, \quad \frac{\partial A_0}{\partial z}\,|_{z=0}=\frac{1}{\mu_m}\frac{\partial A_1}{\partial z}\,|_{z=0}. \qquad (5)$$

In addition, we assume that the functions A_0 and A_1 satisfy the following conditions at infinity:

$$A_i,\ \frac{\partial A_i}{\partial r}\to 0, \text{as } r\to\infty,\ i=0,1, \qquad (6)$$
$$A_0\to 0, \text{as } z\to+\infty,\ A_1\to 0, \text{as } z\to-\infty. \qquad (7)$$

In order to solve (3) – (7) we use the Hankel transform of the form

$$\tilde{A}_i(\lambda,z)=\int_0^\infty A_i(r,z)rJ_1(\lambda r)dr, \quad i=0,1, \qquad (8)$$

where $J_1(x)$ is the Bessel function of the first kind of order 1. Applying (8) to problem (3) – (7) we obtain

$$\frac{d^2\tilde{A}_0}{dz^2}-\lambda^2\tilde{A}_0=-\mu_0 Ir_c J_1(\lambda r_c)\delta(z-h), \qquad (9)$$
$$\frac{d^2\tilde{A}_1}{dz^2}-\lambda^2\tilde{A}_1-j\omega\mu_0\mu_m\sigma_m e^{(\alpha+\beta)z}\tilde{A}_1-\beta\frac{d\tilde{A}_1}{dz}=0, \qquad (10)$$
$$\tilde{A}_0\,|_{z=0}=\tilde{A}_1\,|_{z=0}, \quad \frac{d\tilde{A}_0}{dz}\,|_{z=0}=\frac{1}{\mu_m}\frac{d\tilde{A}_1}{dz}\,|_{z=0}, \qquad (11)$$

$$\tilde{A}_0\to 0, \text{as } z\to+\infty,\ \tilde{A}_1\to 0, \text{as } z\to-\infty. \qquad (12)$$

The solution to (9) can be easily obtained in the two regions $0<z<h$ and $z>h$. We denote the solution to (9) in each of the two regions by $\tilde{A}_{00}$ and $\tilde{A}_{01}$, respectively. Thus,

$$\frac{d^2\tilde{A}_{00}}{dz^2}-\lambda^2\tilde{A}_{00}=0, \qquad 0<z<h, \qquad (13)$$
$$\frac{d^2\tilde{A}_{01}}{dz^2}-\lambda^2\tilde{A}_{01}=0, \qquad z>h. \qquad (14)$$

The general solution to (13) is

$$\tilde{A}_{00}=C_1e^{\lambda z}+C_2e^{-\lambda z}. \qquad (15)$$

The solution to (14) which satisfies (12), that is, which is bounded as $z\to+\infty$, has the form

$$\tilde{A}_{01}=C_3e^{-\lambda z}. \qquad (16)$$

The solution to (10) which is bounded as $z\to-\infty$ can be found in terms of the Bessel functions (see [8], formula 2.1.3.10, page 247):

$$\tilde{A}_1(\lambda,z)=C_4e^{\beta z/2}I_\nu\left(ce^{(\alpha+\beta)z/2}\right), \qquad (17)$$

where $c=\frac{2\sqrt{j\omega\mu_0\mu_m\sigma_m}}{\alpha+\beta}$.

Using (15) and (16) and the fact that the functions $\tilde{A}_{00}$ and $\tilde{A}_{01}$ are continuous at $z=h$ we obtain

$$C_1e^{\lambda h}+C_2e^{-\lambda h}=C_3e^{-\lambda h}. \qquad (18)$$

We integrate (9) with respect to z from $h-\varepsilon$ to $h+\varepsilon$:

$$\left.\frac{d\tilde{A}_0}{dz}\right|_{h-\varepsilon}^{h+\varepsilon}-2\varepsilon\lambda^2\tilde{A}_0(\lambda,\xi)=-\mu_0 Ir_cJ_1(\lambda r_c), \qquad (19)$$

where ξ is a point in the interval $h-\varepsilon<\xi<h+\varepsilon$.

The main property of the delta-function and the mean value theorem is used to derive (19). Taking the limit of (19) as $\varepsilon\to+0$ we obtain

$$\left.\frac{d\tilde{A}_{01}}{dz}\right|_{z=h}-\left.\frac{d\tilde{A}_{00}}{dz}\right|_{z=h}=-\mu_0 Ir_cJ_1(\lambda r_c). \qquad (20)$$

Using (15), (16) and (20) we obtain

$$-C_3\lambda e^{-\lambda h}-C_1\lambda e^{\lambda h}+C_2\lambda e^{-\lambda h}=-\mu_0 Ir_cJ_1(\lambda r_c).$$

Two equations for the unknown constants C_1, C_2 and C_4 are obtained from (11) and have the form

$$C_1+C_2=C_4I_\nu(c), \qquad (21)$$

$$C_1 - C_2 = \frac{C_4}{\lambda\mu_m}\left[\frac{\beta}{2} I_\nu(c) + \frac{c(\alpha+\beta)}{2} I_\nu^{'}(c)\right]. \quad (22)$$

The unknown constants C_1, C_2, C_3 and C_4 are obtained from the solution of the linear system (18), (20) – (22). In particular, the values of the constants C_2 and C_4 are

$$C_2 = \frac{\mu_0 I r_c J_1(\lambda r_c) e^{-\lambda h}[(2\lambda\mu_m - \beta) I_\nu(c) - c(\alpha+\beta) I_\nu^{'}(c)]}{2\lambda[(2\lambda\mu_m + \beta) I_\nu(c) + c(\alpha+\beta) I_\nu^{'}(c)]}, \quad (23)$$

$$C_4 = \frac{2\mu_0 \mu_m I r_c J_1(\lambda r_c) e^{-\lambda h}}{(2\lambda\mu_m + \beta) I_\nu(c) + c(\alpha+\beta) I_\nu^{'}(c)}. \quad (24)$$

It can be shown (see [7]) that the induced vector potential $\tilde{A}_0^{ind}(\lambda, z)$ is given by

$$\tilde{A}_0^{ind}(\lambda, z) = C_2 e^{-\lambda z}, \quad (25)$$

where C_2 is calculated by means of (23). Applying the inverse Hankel transform

$$A_i(r,z) = \int_0^\infty \tilde{A}_i(\lambda, z)\lambda J_1(\lambda r) d\lambda, \quad i = 0,1 \quad (26)$$

to (25) we obtain the induced vector potential in the form

$$A_0^{ind}(r,z) = \frac{\mu_0 I r_c}{2}\int_0^\infty F(\lambda) J_1(\lambda r_c) J_1(\lambda r) e^{-\lambda(z+h)} d\lambda, \quad (27)$$

where

$$F(\lambda) = \frac{(2\lambda\mu_m - \beta) I_\nu(c) - c(\alpha+\beta) I_\nu^{'}(c)}{(2\lambda\mu_m - \beta) I_\nu(c) + c(\alpha+\beta) I_\nu^{'}(c)}. \quad (28)$$

The induced change in impedance, Z^{ind}, is obtained from the formula (see [7]):

$$Z^{ind} = \frac{j\omega}{I} A_0^{ind}(r_c, h) \cdot 2\pi r_c. \quad (29)$$

Using (27) and (29) we obtain the induced change in impedance in the form

$$Z^{ind} = \pi j \omega \mu_0 r_c^2 \int_0^\infty F(\lambda) J_1^2(\lambda r_c) e^{-2\lambda h} d\lambda. \quad (30)$$

Introducing the dimensionless variable $s = \lambda r_c$ we rewrite formula (30) in the following form

$$Z^{ind} = \pi\omega\mu_0 r_c Z,$$

where

$$Z = j\int_0^\infty \frac{(2s\mu_m - \tilde{\beta}) I_\nu(\tilde{c}) - \tilde{c}(\tilde{\alpha}+\tilde{\beta}) I_\nu^{'}(\tilde{c})}{(2s\mu_m + \tilde{\beta}) I_\nu(\tilde{c}) + \tilde{c}(\tilde{\alpha}+\tilde{\beta}) I_\nu^{'}(\tilde{c})} J_1^2(s) e^{-2\gamma s} ds. \quad (31)$$

The following notations are used in (31):

$$\tilde{c} = \frac{2\eta\sqrt{j}}{\tilde{\alpha}+\tilde{\beta}}, \ \nu = \frac{\sqrt{\tilde{\beta}^2 + 4s^2}}{\tilde{\alpha}+\tilde{\beta}},$$

$$\eta = r_c\sqrt{\omega\sigma_m\mu_0\mu_m}, \ \tilde{\alpha} = \alpha r_c, \ \tilde{\beta} = \beta r_c, \ \gamma = \frac{h}{r_c}.$$

If electric and magnetic properties of a conducting half-space are constants ($\tilde{\alpha} = 0, \tilde{\beta} = 0$) then the solution for the change in impedance has the form (see [7]):

$$Z^{ind} = \pi\omega\mu_0 r_c Z,$$

where

$$Z = j\int_0^\infty \frac{s\mu_m - \sqrt{s^2 + j\eta^2}}{s\mu_m + \sqrt{s^2 + j\eta^2}} J_1^2(s) e^{-2\gamma s} ds. \quad (32)$$

Coil of finite dimensions above a half-space with depth-varying electric and magnetic properties

Consider the coil located at a distance h_1 above a conducting half-space where σ and μ vary with depth as in (1). The height of the coil is $h_2 - h_1$ and the inner and outer radii are r_1 and r_2, respectively. Let w be the number of turns in the coil. Consider two rings in the coil, centered at the points (r_n, z_n) and (r_m, z_m), respectively. The vector potential on the contour of the ring centered at (r_m, z_m) due to eddy currents induced in the ring centered at (r_n, z_n) is

$$A_{mn}^{ind} = \frac{\mu_0 I r_n}{2} \cdot \frac{w dz dr}{(h_2 - h_1)(r_2 - r_1)} \int_0^\infty F(\lambda) J_1(\lambda r_n) J_1(\lambda r_m) e^{-\lambda(z_n + z_m)} d\lambda, \quad (33)$$

where $\frac{w dz dr}{(h_2 - h_1)(r_2 - r_1)}$ is the number of turns in the ring centered at (r_n, z_n). Integrating (33) with respect to r_n from r_1 to r_2 and with respect to z_n from h_1 to h_2 we obtain the vector potential in the ring centered at (r_m, z_m) due to eddy currents induced by the whole coil:

$$A_m^{ind} = -\frac{\mu_0 I w}{2(h_2 - h_1)(r_2 - r_1)} \int_0^\infty \frac{F(\lambda)}{\lambda} J_1(\lambda r_m) e^{-\lambda z_m} \left[\int_{r_1}^{r_2} r_n J_1(\lambda r_n) dr_n\right] \frac{e^{-\lambda h_1}}{\lambda} \left[e^{-\lambda(h_2 - h_1)} - 1\right] d\lambda. \quad (34)$$

Integrating (34) with respect to r_m from r_1 to r_2 and with respect to z_m from h_1 to h_2 we obtain the induced vector potential of the coil in the form

$$A_{coil}^{ind} = \frac{\mu_0 I w^2}{2(h_2-h_1)^2(r_2-r_1)^2}\int_0^\infty \frac{F(\lambda)}{\lambda^6}\left(e^{-\lambda h_2}-e^{-\lambda h_1}\right)^2 \eta^2(r_1,r_2,\lambda)d\lambda, \tag{35}$$

where $\eta = \int_{\lambda r_1}^{\lambda r_2} \xi J_1(\xi)d\xi$.

The induced change in impedance can be written in the form

$$Z_{coil}^{ind} = j\omega\pi\mu_0 \frac{w^2}{(h_2-h_1)^2(r_2-r_1)^2}\int_0^\infty \frac{F(\lambda)}{\lambda^6}\left(e^{-\lambda h_2}-e^{-\lambda h_1}\right)^2 \eta^2(r_1,r_2,\lambda)d\lambda \cdot \tag{34}$$

Numerical results

The impedance change Z, computed by means of (31) and (32) is shown in Fig. 1 for the case $\gamma = 0.05$, $\mu_m = 5$, $\tilde{\alpha} = 0$ and different values of $\tilde{\beta}$. Calculations are done by means of Mathematica. The choice of the software package depends on the complexity of the problem. Mathematica can be effectively used to compute integrals (31) and (32). First, Mathematica has built-in routine to compute definite integrals. Second, it also has an option of calculating Bessel functions of variable order and complex argument.

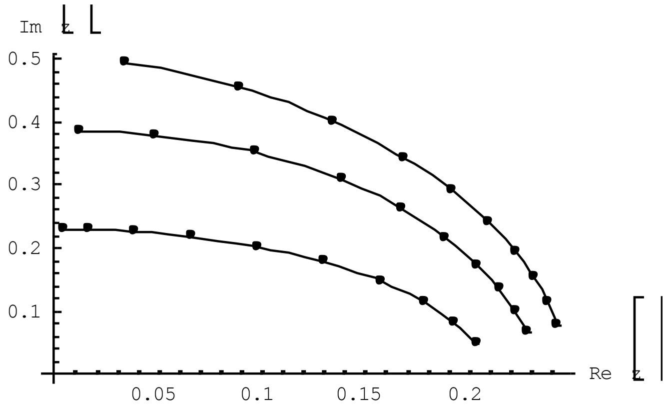

Fig. 1. The change in impedance of the single-turn coil as a function of η due to the presence of a conducting half-space with variable magnetic properties. The curves (from top to bottom) correspond to the cases $\tilde{\beta} = 5, \tilde{\beta} = 2$ and $\tilde{\beta} = 0$, respectively

The calculated points shown in Fig. 1 correspond to the values of $\eta = 1, 2, ...10$ (from left to right). As can be seen from the graph, the increase in $\tilde{\beta}$ leads to the larger values of the components of the induced change in impedance (both real and imaginary parts increase for the same value of η).

Conclusions

The formulas for the change in impedance of a single-turn coil and coil of finite dimensions for the case where the coils are located above a conducting half-space whose magnetic permeability and electric conductivity are exponential functions of the vertical coordinate are obtained in the paper. The solution of the problem for the vector potential of the coils is obtained in closed form in terms of improper integrals containing Bessel functions. Results of numerical calculations are presented.

References

1. Cheng-Chi Tai, Hung-Chi Yang, Yin-Hsiung Liu. Modeling the surface condition of ferromagnetic metal by the swept-frequency eddy current method. *IEEE Transactions on Magnetics*, 38:205–210, 2002.
2. Cheng-Chi Tai. Characterization of coatings on magnetic metal using the swept-frequency eddy current method *Review of Scientific Instruments*, 71:3161–3167, 2000.
3. C. V. Dodd, and W. E. Deeds. Analytical solutions to eddy-current probe-coil problems. *Journal of Applied Physics*, 39: 2829–2838, 1968.
4. E. Uzal and J.H. Rose. The impedance of eddy current probes above layered metals whose conductivity and permeability vary continuously. *IEEE Transactions on Magnetics*, 29: 1869–1873, 1993.
5. G.A. Kasimov and and Yu.V. Kulaev. An applied electromagnetic transducer above the object of inspection with the electrical and magnetic properties of the material changing with depth. *The Soviet Journal of Nondestructive Testing*, 14: 550–552, 1978.
6. E. Uzal, J.C. Moulder, S. Mitra, and J.H. Rose. Impedance of coils over layered metals with continuously variable conductivity and permeability: Theory and experiment. *Journal of Applied Physics*, 74: 2076–2089, 1993.
7. M.Ya Antimirov, A.A. Kolyshkin, R. Vaillancourt. *Mathematical models for eddy current testing*. CRM, Montreal, Canada, 1997.
8. A.D. Polyanin, and V.F Zaitsev. *Handbook of exact solutions for ordinary differential equations*. Chapman & Hall/CRC, Florida, USA, 2003.

Valentina Koliskina, M.Sc.
Riga Technical University, Department of Engineering Mathematics
Address: 1 Meza street, block 4, Riga, LV1048, Latvia
Phone: 371-6770-89528
Email: v.koliskina@gmail.com

Inta Volodko, Ph.D.
Riga Technical University, Department of Engineering Mathematics
Address: 1 Meza street, block 4, Riga, LV1048, Latvia
Phone: 371-6770-89528
Email: inta.volodko@rtu.lv

Koliškina V., Volodko I. Spoles pretestība virs elektriski vadošas pustelpas ar mainīgām elektriskām un magnētiskām īpašībām
Rakstā ir iegūta divu parametru analītisko atrisinājumu saime uzdevumam par vijuma ar mainīgo strāvu elektromagnētisko lauku virs elektriski vadošas pustelpas. Vides elektriskā vadāmība un magnētiskā caurlaidība ir eksponenciālas funkcijas no vertikālās koordinātas. Problēma ir atrisināta ar Hankeļa integrālo transformācijas metodi. Atrisinājums ir iegūts ar neīsto integrāli, kas satur Beseļa funkcijas. Ir iegūti arī skaitliskie rezultāti. Iegūtais atrisinājums ir vispārināts galīgo izmēru spoles gadījumam. Rakstā iegūtās formulas var izmantot inversās problēmas atrisinājumam (pustelpas parametru noteikšanai) gadījumos, kad vides elektriskā vadāmība un magnētiskā caurlaidība nav konstanti lielumi.

Колышкина В., Володко И. Импеданс катушки над полупространством с переменными электрическими и магнитными свойствами
В статье получено двухпараметрическое семейство аналитических решений задачи о поле витка с переменным током, расположенного над проводящим полупространством. Электрическая проводимость и магнитная проницаемость полупространства являются экспоненциальными функциями от вертикальной координаты. Задача решена с помощью интегрального преобразования Ханкеля. Решение получено в виде несобственных интегралов, содержащих функции Бесселя. Приведены результаты расчетов по полученным формулам. Полученное решение обобщено также на случай катушки конечных размеров. Полученные в статье формулы могут быть использованы для решения обратной задачи определения параметров проводящего полупространства в случаях, когда электропроводность и магнитная проницаемость среды не являются постоянными величинами.